SECTION 9. MECHANICAL ENGINEERING

9.1 Algorithm for calculating the vibration resistance of shafts of mechanical mixing devices

INTODUCTION

A significant amount of damage to parts of modern machines and their structures is due to stresses arising from oscillations. Vibration load can cause damage, especially in the formation of resonant states. Shafts of mechanical mixing devices (MMDs) are exposed to this kind of danger. Therefore, when designing them, it is important to correctly choose the design schemes for the various options for securing the shaft in bearings.

Depending on the location of the bearings, the shafts are divided into single-span and cantilever. Most often, the shafts are considered as straight statically defined rods, fixed in hinged supports and subjected to deflection (bending) by specified stresses. The accuracy of determining the diameters of vibration-resistant shafts is determined by methods of taking into account the geometric and physical parameters of the shaft: size, weight, location of supports, the number of stirrers and points of attachment to the shaft.

The best results are given by calculation of shafts on vibration resistance method by acad. Krylov O.M., which, however, is extremely difficult if the number of concentrated masses on the shaft is more than one. Therefore, for engineering calculations use the method of reduction, developed by acad. Szymanski Yu. O. Calculation by the method of reduction is performed by successive approximations, which is greatly facilitated by the use of computers.

The monograph considers the algorithm for calculating the vibration resistance of rigid single-span shafts, rigid and flexible cantilever smooth shafts, rigid cantilever step MMD shafts. To solve such an algorithm needed:

- to study (know) the design of the MMDs and the method of mounting the shaft in the supports;

- to justify the choice of calculation scheme and make it;
- to make an algorithm for calculating the vibration resistance of the MMD shaft;
- to understand and use a computer program according to the calculation algorithm and model

9.1.1 Terms and definitions

The problem of calculating the MMD shafts consists of choosing such modes of its operation and such geometric parameters at which the angular frequency (angular velocity) of the forced oscillations of the shaft ω would be in the right ratio with the frequency of its natural oscillations.

Intrinsic (free) are oscillations that occur in an isolated system due to external excitation (shock), which causes at the points of the system initial deviations from equilibrium or initial velocities and those that continue due to the presence of internal elastic forces that establish equilibrium.

Consider a MMD shaft having one pivot resistance at point A in the form of a ball bearing that receives radial and axial stresses, and a pivot bearing at point C in the form of a ball bearing that receives radial stress and that can be displaced in the axial direction (Fig. 1, a-c). An example of the natural oscillations of such a shaft will be transverse (bending) oscillations, if the mass m is applied a force F in the form of a shock in the direction perpendicular to the shaft.

Since the energy that caused (excited) the oscillation process came from the outside only at the initial moment, the nature of the natural oscillations is determined by the internal elastic forces of the shaft, which depend on its physical and geometric structure. The amplitude of such oscillations depends on the shock force, and the frequency, which is called the **natural oscillation frequency**, depends only on the properties of the shaft. Thus, the natural frequency of the system, which is determined by the physical structure and geometric parameters, should also be considered as a property or characteristic of the system.

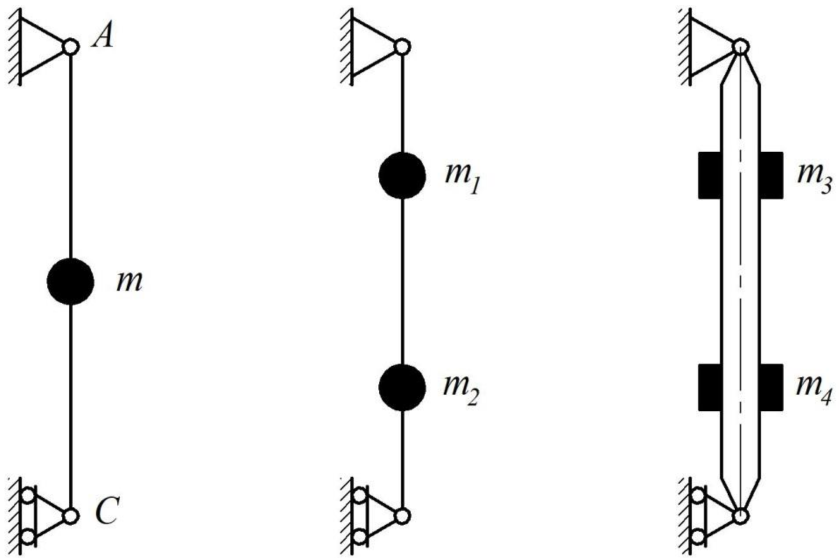


Fig. 1 Scheme to determine the number of degrees of freedom of the oscillating system (MMD shaft)

Oscillations of an elastic system that occur during continuous action on a system of given external, periodically changing perturbing forces acting independently of oscillations in the system are called **forced**. The parameters of the process of forced oscillations depend on both the properties of the system and external forces.

Any elastic system that oscillates at any given time is located in space, which is determined by the number of independent parameters of the oscillating system is called the **number of degrees of freedom**. Thus, a shaft whose mass can be neglected in relation to the concentrated mass m (Fig. 1, a) has one degree of freedom. The same shaft, but with masses m_1 and m_2 (see Fig. 1, b), has two degrees of freedom, because the parameters mentioned here - the movement of masses m_1 and m_2 relative to the equilibrium position. A shaft whose mass cannot be neglected in comparison with the masses m_3 and m_4 (see Fig. 1, c) is a system with an infinite number of degrees of freedom [478].

9.1.2 Guidelines for calculating the natural oscillation frequency of a weightless MMD shaft with one degree of freedom

A beam loaded with the same forces and having the same supports as the shaft can be taken as the design scheme of a shaft with a mixing device (Fig. 2).

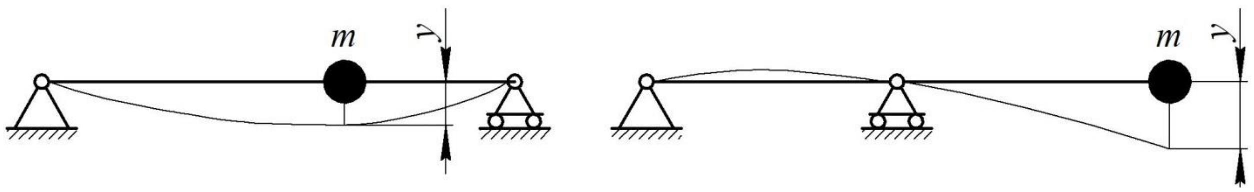


Fig. 2 Calculation scheme: a – single-span shaft; b – cantilever shaft

In the absence of friction forces for any shaft structure with one degree of freedom, the differential equation of its oscillating motion (bending oscillation) has the form

$$m \cdot \ddot{y} + k \cdot y = 0 \tag{1}$$

Where *m* is the concentrated mass of the cargo; $\ddot{y}$ is the second derivative of mass displacement *m* in time (acceleration); *k* is the stiffness coefficient of the shaft, which is a static force that causes a single displacement of the mass *m*; *y* is the coordinate of the oscillating mass subtracted from its mean position.

Denoting ω_c^2=k/m we rewrite the differential equation (1):

$$\ddot{y} + \omega_c^2 \cdot y = 0$$

The solution of which is the following equation

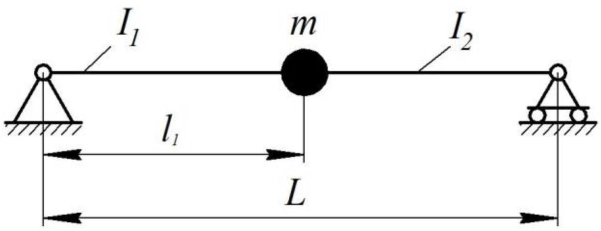
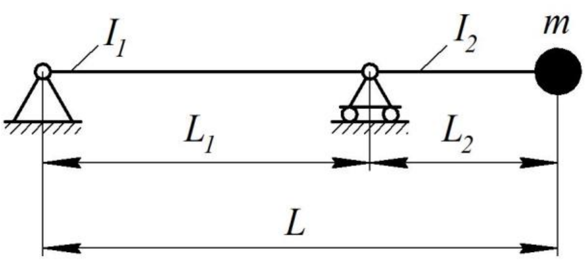
$$y = A\sin(\omega_c t + \varphi)\,.$$

Where *A* is the amplitude of oscillations; the expression $\omega_c t+\phi$ is called **the phase of oscillations**, and ϕ is **the phase shift**. Angular frequency of natural oscillations of the shaft, *rad/s* (s^{-1}):

$$\omega_c = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{(m \cdot \delta_{11})}} \tag{2}$$

Where δ_{11} is the displacement of the mass m (**deflection of the shaft**) from the unit transverse force applied at the point of attachment of the mass *m*, or **the coefficient of influence**. In the table. 1 shown the calculated dependences for determining the coefficients of influence δ_{11} of weightless shafts with one degree of freedom [479].

Table 1 Coefficient of influence δ_{11} for shafts with one concentrated mass

Settlement schemes	Estimated dependencies
	$\delta_{11} = \frac{l_1^3 (L - l_1)^2}{3E \cdot I_1 \cdot L^2} + \frac{l_1^3 (L - l_1)^3}{3E \cdot I_2 \cdot L^2}$ For $I_1 = I_2 = I = \frac{\pi \cdot d^4}{64}$ $\delta_{11} = \frac{l_1^2 (L - l_1)^2}{3E \cdot I \cdot L}$
	$\delta_{11} = \frac{L \cdot L_1^2}{3E \cdot I_1} + \frac{L_1^3}{3E \cdot I_2}$ For $I_1 = I_2 = I = \frac{\pi \cdot d^4}{64}$ $\delta_{11} = \frac{L_2 \cdot L_1^2}{3E \cdot I}$

Knowing the frequency of natural oscillations, you can find **the period of natural oscillations** (time of one complete oscillation):

$$T = \frac{2\pi}{\omega_c} = 2\pi\sqrt{m \cdot \delta_{11}}$$

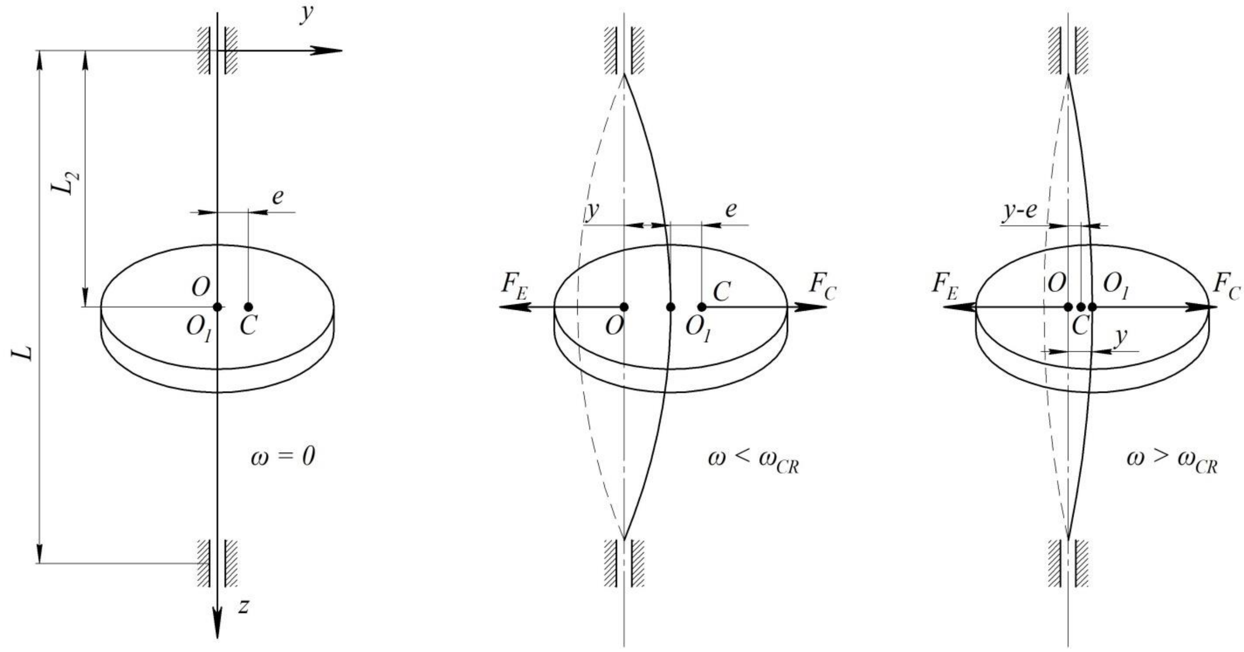
The value inverse to the period of oscillations determines the number of oscillations per unit time (second) and is called **the second frequency**

$$f = \frac{1}{T} = \frac{\omega_c}{2\pi}.$$

9.1.3 Instructions for calculating the critical angular velocity and vibration resistance condition of a smooth weightless shaft with one degree of freedom

The practice of operation of machines, including devices with MMD, shows that rotating shafts at specific angular speeds for these machines, falling into resonance, become dynamically unstable: there are large transverse oscillations (deflections) of the shafts and even the possibility of their destruction. The angular velocity of rotation of the shaft at which the resonance phenomenon occurs is called **the critical angular**

velocity ω_{CR}. To find it, consider a two-support vertical shaft (Fig. 3, a), in the middle of which is a disk of mass *m*, planted with eccentricity *e* (when assembling a disk with a shaft, the eccentricity of landing is almost inevitable).



a b c

Fig.3. Scheme of the center of inertia of the disk (point C), fixed inside a single-span shaft: a – non-rotating; b and c – rotating according to the angular velocity $\omega < \omega_{CR}$ and $\omega > \omega_{CR}$

In Fig.3 point C corresponds to the center of inertia of the disk, point O – the intersection of the bearing axis with the middle surface of the disk. When the shaft rotates around its axis with an angular velocity ω, its axis bends under the action of the centrifugal force F_c on the value of *y* (Fig. 3, b). The curved axis of the shaft rotates around the axis of the bearings with the same angular velocity *ω*, such rotation of the shaft is called **direct synchronous precession**.

The centrifugal force applied in the center of inertia of the disk:

$$F_C = m \cdot \omega^2 (y + e)$$

In the opposite direction from the side of the shaft on the disk acts **the force of elasticity** applied at point O_1:

$$F_E = K \cdot y = \frac{y}{\delta_{11}}$$

Where the shaft stiffness coefficient at the symmetrical arrangement of the disk $K=1/\delta_{11}=48EI/L^3$ (E - modulus of longitudinal elasticity of the shaft material); I, L - respectively, the moment of inertia of the cross section of the shaft and its length.

Equating F_C and F_E, we get

$$m \cdot \omega^2 (y + e) = \frac{y}{\delta_{11}},$$

Whereof

$$y = \frac{\omega^2 \cdot l}{\frac{1}{m \cdot \delta_{11}} - \omega^2}.$$

Given that $1/(m \cdot \delta_{11})$ is the square of the angular frequency ω_c of free oscillations, we rewrite the last formula in the form

$$y = \frac{l}{\frac{\omega_c^2}{\omega^2} - 1}.$$

From the obtained equality it follows that for $\omega_c/\omega \to 1$ the deflection of the shaft $y \to \infty$. Hence the conclusion that the critical angular velocity of the rotating shaft is equal to the angular frequency of its natural oscillations:

$$\omega_{CR} = \omega_c = \sqrt{\frac{1}{m \cdot \delta_{11}}}.$$

The above calculations are made without taking into account the resistance to rotation of the environment. In real devices, when energy is lost to overcome this resistance, the shaft's resonance does not always lead to destructive deflections.

It is known from experiments that at $\omega > (\omega_{CR} = \omega_c)$ the deflections of the shaft begin to decrease and the center of inertia of the disk C is located between the axis of the bearings and the curved axis of the shaft (Fig. 3, b - between points O and O_1). When the disk rotates at supercritical speed, its centrifugal force of inertia

$$F_C = m \cdot \omega^2 (y - e).$$

Equating F_y to the force of elasticity of the shaft F_E and solving the obtained equation with respect to y, we find

$$y = \frac{l}{1 - \frac{\omega_c^2}{\omega^2}}.$$

As can be seen from the last formula, when the shaft rotates at a speed much higher than the critical one $(\omega \geq \omega_c)$, $y \to e$, and the shaft becomes dynamically stable again.

Restoration of shaft stability at supercritical speed occurs as a result of Coriolis acceleration at the moment when the center of inertia of the disk begins to move in the radial direction from point C. Under Coriolis forces point O_1 begins to move in the direction perpendicular to the radius, and eventually comes to a position on the other side of point C.

If the working speed of the shaft is less than critical, the shaft is called **rigid**, otherwise - **flexible**. The working speed of the shaft ω should be significantly different from the critical, and the condition of vibration resistance of the shaft is:

For rigid shaft

$$\omega \leq 0{,}7\omega_{CR};$$

For flexible shaft

$$\omega \geq 1{,}3\omega_{CR}.$$

When designing flexible shafts provide rapid passage of critical speed in order to reduce resonant deflections. This, however, requires the installation of high-power engines that provide rapid acceleration and powerful brakes that stop the shaft abruptly.

9.1.4 Instructions for calculating the natural oscillation frequency of a smooth weightless shaft with two or more degrees of freedom

Consider a system that is a shaft in section 1 of which is a disk of mass m_1, and in section 2 - a disk of mass m_2 (Fig. 4). Due to the eccentricities e_1 and e_2 of the disks on the shaft when the system rotates with an angular velocity ω, centrifugal forces F_{C1} and F_{C2} occur, causing the shaft to bend.

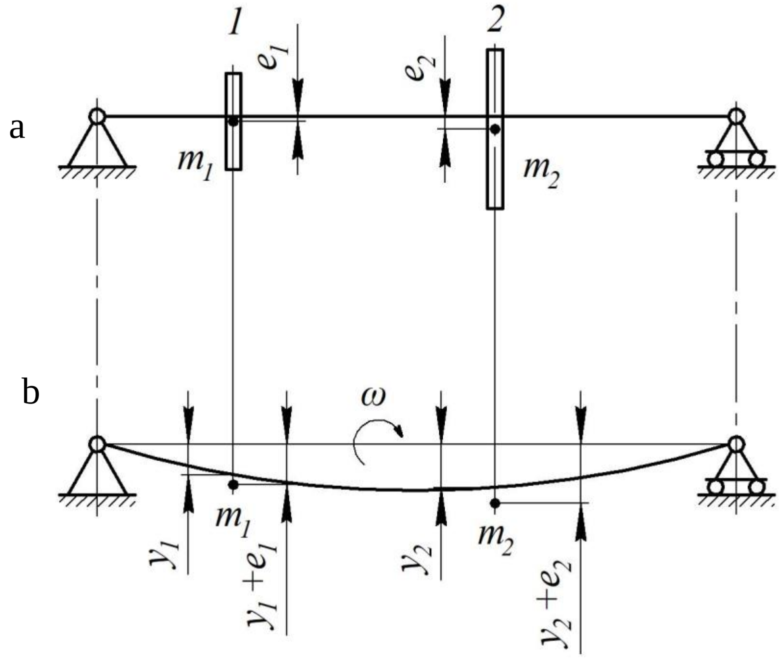


Fig. 4 Scheme of a horizontal shaft with two disks: a – without rotating; b – at rotating

From the condition of elastic equilibrium of the curved shaft we determine the deflections (bends) y_1 in section 1 and y_2 in section 2 from the action of centrifugal forces:

$$\begin{cases} y_1 = \delta_{11} \cdot F_{C1} + \delta_{12} \cdot F_{C2}; \\ y_2 = \delta_{21} \cdot F_{C1} + \delta_{22} \cdot F_{C2}. \end{cases} \tag{3}$$

Where δ_{11}, δ_{12} – deflection (coefficient of influence) in section 1 of the unit centrifugal force caused by the rotation of the mass m_1 and m_2, respectively; δ_{21}, δ_{22} – deflection in section 2 of the unit centrifugal force caused by the rotation of the mass m_1 and m_2, respectively. As is known from the reciprocity theorem $\delta_{12}=\delta_{21}$. Substitute the values of F_{C1} and F_{C2} into system (3) and write them down

$$\begin{cases} y_1 = \delta_{11} \cdot m_1 \cdot \omega^2 (e_1 + y_1) + \delta_{12} \cdot m_2 \cdot \omega^2 (e_2 + y_2) \\ y_2 = \delta_{21} \cdot m_1 \cdot \omega^2 (e_1 + y_1) + \delta_{22} \cdot m_2 \cdot \omega^2 (e_2 + y_2) \end{cases}.$$

Grouping members with unknowns y_1 and y_2, we finally get

$$\begin{cases} y_1(\delta_{11} \cdot m_1 \cdot \omega^2 - 1) + y_2 \cdot \delta_{12} \cdot m_2 \cdot \omega^2 = -\omega^2 (\delta_{11} \cdot m_1 \cdot e_1 - \delta_{12} \cdot m_2 \cdot e_2) \\ y_1 \cdot \delta_{21} \cdot m_1 \cdot \omega^2 + y_2 (\delta_{22} \cdot m_2 \cdot \omega^2 - 1) = -\omega^2 (\delta_{21} \cdot m_1 \cdot e_1 - \delta_{22} \cdot m_2 \cdot e_2) \end{cases}.$$

In order for a system of n linear homogeneous equations with n unknowns to have non-zero roots, it is necessary that the determinant consisting of the coefficients for the unknowns in the system of equations be equal to zero. Write the determinant, which consists of the coefficients y_1 and y_2 and equate it to zero:

$$\begin{vmatrix} \delta_{11} \cdot m_1 \cdot \omega^2 - 1 & \delta_{12} \cdot m_2 \cdot \omega^2 \\ \delta_{21} \cdot m_1 \cdot \omega^2 & \delta_{22} \cdot m_2 \cdot \omega^2 - 1 \end{vmatrix} = 0 .$$

We open the determinant and obtain the frequency equation corresponding to the case when $\omega = \omega_c = \omega_{cr}$ and the deflections of the shaft increase infinitely:

$$\left(\delta_{11} \cdot m_1 \cdot \omega_{CR}^2 - 1\right)\left(\delta_{22} \cdot m_2 \cdot \omega_{CR}^2 - 1\right) - \delta_{12}^2 \cdot m_1 \cdot m_2 \cdot \omega_{CR}^4 = 0 , \tag{4}$$

Whence we define two values of critical angular speed of a shaft (on physical maintenance of a problem we take only positive roots):

$$\omega_{CR1} = \sqrt{\frac{1}{2\left(\delta_{11} \cdot \delta_{22} - \delta_{12}^2\right)m_2}\left[\delta_{11} + \delta_{22}\frac{m_2}{m_1} + \sqrt{\left(\delta_{11} + \delta_{22}\frac{m_2}{m_1}\right)^2 - 4\left(\delta_{11} \cdot \delta_{22} - \delta_{12}^2\right)\frac{m_2}{m_1}}\right]} ;$$

$$\omega_{CR2} = \sqrt{\frac{1}{2\left(\delta_{11} \cdot \delta_{22} - \delta_{12}^2\right)m_2}\left[\delta_{11} + \delta_{22}\frac{m_2}{m_1} - \sqrt{\left(\delta_{11} + \delta_{22}\frac{m_2}{m_1}\right)^2 - 4\left(\delta_{11} \cdot \delta_{22} - \delta_{12}^2\right)\frac{m_2}{m_1}}\right]} .$$

As we can see, a system rotating with two degrees of freedom (two concentrated masses) has two critical angular velocities. Similarly, it can be shown for the general case that the number of critical angular velocities is equal to the number of masses fixed on the weightless shaft. However, to determine the critical angular velocities with a degree of freedom greater than two, this method is unacceptable due to the complexity of the calculations. As a method of approximations (approximate) determination of the first critical angular velocity ω_{CR1} of the system with more than one degree of freedom can be calculated by the Donkerley formula:

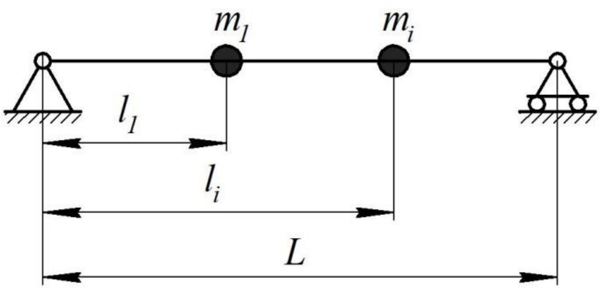
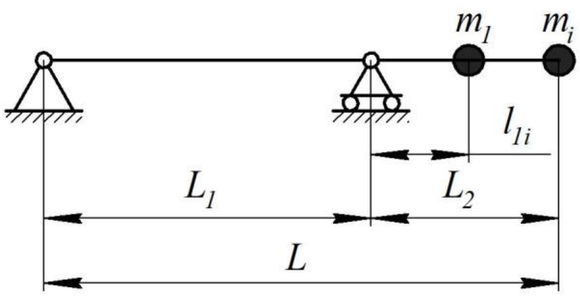
$$\frac{1}{\omega_{CR1}^2} = \frac{1}{\omega_{1CR1}^2} + \frac{1}{\omega_{2CR1}^2} + \frac{1}{\omega_{3CR1}^2} + \ldots + \frac{1}{\omega_{iCR1}^2} , \tag{5}$$

Where

$$\omega_{1CR1} = \sqrt{\frac{1}{m_1 \cdot \delta_{11}}} ,\ \omega_{2CR1} = \sqrt{\frac{1}{m_2 \cdot \delta_{22}}} , \ldots ,\ \omega_{iCR1} = \sqrt{\frac{1}{m_i \cdot \delta_{ii}}} .$$

Here δ_{11}, δ_{12},…, δ_{ii} – coefficients of influence at installation on a shaft of one i-th disk in the i-th section, their values for single-span and cantilever shafts are given in Table 2 [479].

Table 2 Coefficients of influence δ_{11},…, δ_{ii} for weightless shafts with i-th degrees of freedom

Settlement schemes	Estimated dependencies
	$\delta_{11}=\frac{l_1^2(L-l_1)^2}{3E\cdot I\cdot L};...;\delta_{ii}=\frac{l_i^2(L-l_i)^2}{3E\cdot I\cdot L}$
	$\delta_{11}=\frac{L_2\cdot L_1^2}{3E\cdot I};...;\delta_{ii}=\frac{(L+l_{1i})l_{1i}^2}{3E\cdot I}$

Formulas (2), (4), (5) give only fairly approximate values of the critical angular velocities of the MMD shafts. To avoid destructive resonant phenomena in the calculations it is necessary to take into account the own weight of the shaft.

9.1.5 Instructions for calculating the frequency of natural oscillations of a smooth shaft, taking into account its uniformly distributed mass along the length

Let a smooth single-span shaft of length *L* rotate in bearings with angular velocity ω (Fig. 5). Take the mass of a unit of its length m_L, then the continuously distributed centrifugal force of inertia, the mass of the rotating shaft is defined as $m_L\,\omega^2 y$, where *y* is the deflection of the shaft in the *Z* coordinate [480, 481]. This force is balanced by the continuously distributed elastic reaction *q*. From the course of resistance of materials it is known that when bending the rods of a constant length of the section there are relations:

$$D=\frac{dy}{dz};\ M=\frac{d^2y}{dz^2}E\cdot I;\ Q=\frac{d^3y}{dz^3}E\cdot I;\ q=\frac{d^4y}{dz^4}E\cdot I;$$

Where *D, M, Q* and *q* are the angle of rotation of the section, bending moment, transverse force and continuously distributed load in the section with the coordinate *Z*, respectively. Taking into account *q*, we write the equilibrium condition of the rotating shaft

$$E \cdot I \frac{d^4 y}{dz^4} = m_L \cdot \omega^2 \cdot y \, .$$

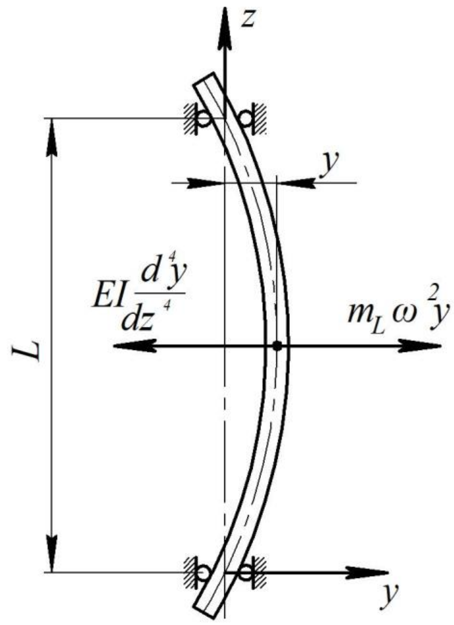


Fig. 5 Scheme of calculation taking into account the influence of the own weight of the shaft on its critical speed

Denoting $m_L\omega^2/EI=\alpha$, we obtain

$$\frac{d^4 y}{dz^4} - \alpha^4 y = 0 \, . \tag{6}$$

Dependence (6) is a linear homogeneous differential equation of the fourth order with constant coefficients. To find its general solution we will make the characteristic equation

$$K^4 - \alpha^4 = 0 \, . \tag{7}$$

Each real root r_i corresponds to the solution e^{r1z} of equation (6). If there are complex roots (they can only be conjugate in pairs, for example, $r_1=a+b_i$; $r_2=a-b_i$), then in the corresponding terms of the general equation the functions e^{r1z} and e^{r2z} should be replaced by $e^{az}cosbz$ and $e^{az}sinbz$. For equation (6) the roots of characteristic equation (7) will be

$$K_1 = \alpha\,; K_2 = -\alpha\,;\; K_3 = \alpha i\,;\; K_4 = -\alpha i\,.$$

Then the general solution of equation (6) will take the form

$$y = A_1 \cdot e^{\alpha z} + A_2 \cdot e^{-\alpha z} + A_3 \cos \alpha z + A_4 \sin \alpha z\,. \tag{8}$$

The constants A_1, A_2, A_3, A_4 are found from the boundary conditions: at $z=0$, $z=4$ (on the supports) the deflections y and the bending moments $M=(d^2y/dz^2)EI$ are zero:

$$\begin{cases} y(0) = 0; y(L) = 0; \\ \dfrac{d^2 y(0)}{dz^2} E \cdot I = 0; \dfrac{d^2 y(L)}{dz^2} E \cdot I = 0. \end{cases} \tag{9}$$

Define the second derivative of function (8)

$$\begin{cases} y^{/} = A_1 \cdot \alpha \cdot e^{\alpha z} - A_2 \cdot \alpha \cdot e^{-\alpha z} - A_3 \cdot \alpha \sin \alpha z + A_4 \cdot \alpha \cdot \sin \alpha z; \\ y^{//} = A_1 \cdot \alpha^2 \cdot e^{\alpha z} + A_2 \cdot \alpha^2 \cdot e^{-\alpha z} - A_3 \cdot \alpha^2 \cos \alpha z - A_4 \cdot \alpha^2 \sin \alpha z. \end{cases} \tag{10}$$

Substitute the boundary conditions (9) in (8) and (10). Since the value of α cannot be zero, we obtain the following system of equations:

$$\begin{cases} A_1 + A_2 + A_3 = 0; \\ A_1 \cdot e^{\alpha L} + A_2 \cdot e^{-\alpha L} + A_3 \cos \alpha L + A_4 \sin \alpha L = 0; \\ A_1 + A_2 - A_3 = 0; \\ A_1 \cdot e^{\alpha L} + A_2 \cdot e^{-\alpha L} - A_3 \cos \alpha L - A_4 \sin \alpha L = 0. \end{cases} \tag{11}$$

Subtracting the third from the first equation of system (11), we have $2A_3=0$, $A_3=0$, where $A_1+A_2=0$; $-A_1=A_2$ and, substituting $A_3=0$, $A_2=-A_1$ in the second and fourth equations, we obtain

$$\begin{cases} A_1 (e^{\alpha L} - e^{-\alpha L}) + A_4 \sin \alpha L = 0; \\ A_1 (e^{\alpha L} - e^{-\alpha L}) - A_4 \sin \alpha L = 0. \end{cases}$$

We make the last equations

$$2A_1 (e^{\alpha L} - e^{-\alpha L}) = 0;\; A_1 = -A_2 = 0\,.$$

When the operating angular velocity ω coincides with one of the critical angular velocities, the deflections in the shaft increase infinitely. Mathematically, this can be done if the determinant consisting of coefficients for homogeneous members of the system of equations (11) is zero.

The solution of the determinant gives $A_4 sin\alpha L=0$. Since $A_4 \neq 0$ (because of what would be $A_1= A_2=A_3=A_4=0$, $y=0$, and the shaft would be at rest), then

$$sin\alpha L=0,$$

That is $\alpha_1 L=\pi$, $\alpha_2 L=2\pi$, ..., $\alpha_i L=i\pi$.

Because $\alpha^4=m_L\omega^2/EI$, then

$$\omega_{CR1} = \alpha_1^2 \sqrt{\frac{E \cdot I}{m_L}} = \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{E \cdot I}{m_L}} ,$$

$$\omega_{CR2} = \alpha_2^2 \sqrt{\frac{E \cdot I}{m_L}} = \left(\frac{2\pi}{L}\right)^2 \sqrt{\frac{E \cdot I}{m_L}} ,$$

....

$$\omega_{CRi} = \alpha_i^2 \sqrt{\frac{E \cdot I}{m_L}} = \left(\frac{i \cdot \pi}{L}\right)^2 \sqrt{\frac{E \cdot I}{m_L}} .$$

As can be seen, a smooth two-support shaft with uniformly distributed mass along the length has an infinite number of natural oscillation frequencies in accordance with the fact that there are an infinite number of degrees of freedom.

Consider a single-span shaft of length L, which carries two concentrated masses with coordinates z_1 and z_2 and has its own mass m_L (Fig. 6). When rotating at an angular velocity ω, the shaft will bend under the action of the centrifugal force $m_L\ \omega^2 y$ (here y is the deflection in the section studied with the z coordinate) and from the concentrated centrifugal forces $F_1(z_1)$ $F_2(z_2)$ caused by the rotation of m_1 and m_2. These forces are balanced by the internal continuously distributed elastic deformation reaction q. Because the formula

$$q = EI\frac{d^4 y}{dz^4},$$

expresses the elastic force of deformation, regardless of the force factors that cause the deflection of the shaft, you can make a differential equation of equilibrium of a rotating shaft and one that has its own uniformly distributed mass and one that carries concentrated masses m_1 and m_2 (4):

$$\frac{d^4 y}{dz^4} = E \cdot I - m_L \cdot \omega^2 \cdot y = F_1(z_1) + F_2(z_2) . \qquad (12)$$

Assuming $\alpha^4=m_L\omega^2/EI$; $f_1(z)=F_1(z_1)/EI$, $f_2(z)=F_2(z_2)/EI$, represent equation (12) in the form

$$\frac{d^4 y}{dz^4} - \alpha \cdot y = f_1(z) + f_2(z) . \qquad (13)$$

The exact analytical solution of the differential equation (13) leads to cumbersome transcendental frequency equations, which are not very suitable for practical application. To find the dependences for determining the roots α of these frequency equations, acceptable in their accuracy for engineering calculations, allows an approximate method based on **the method of reduction**. The essence of this method is to replace a real shaft loaded with distributed and concentrated masses, an oscillating system with a weightless shaft, reduced mass m_{red} and stiffness κ_{red}.

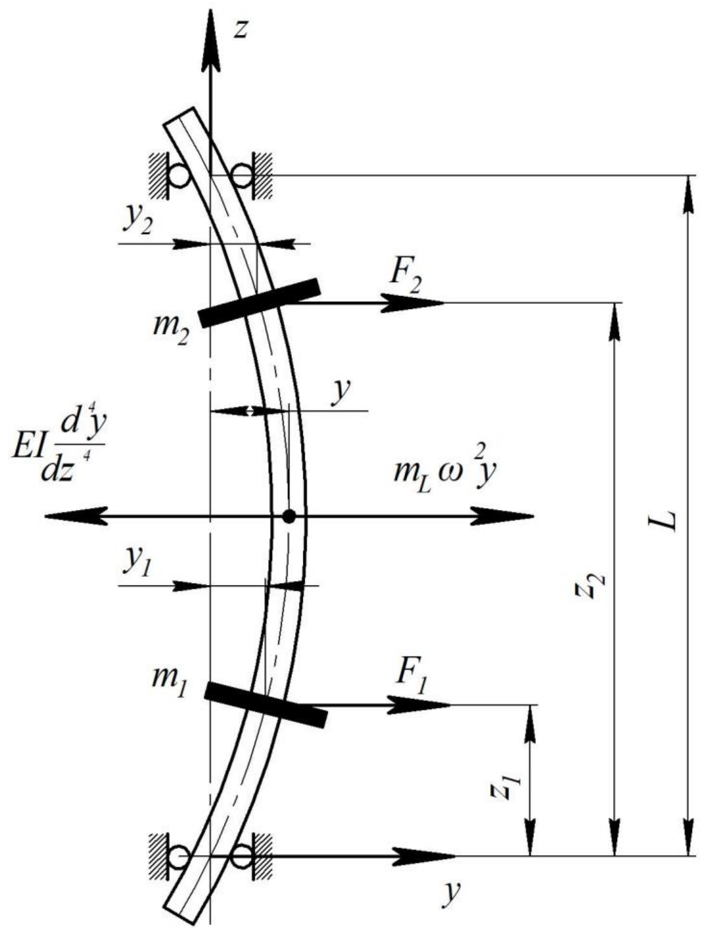


Fig. 6 Scheme of a shaft having an evenly distributed mass with two concentrated masses attached to it

To calculate the shaft of the mixing device, it is necessary to bring its schematic diagram to one of the calculation schemes shown in Fig. 7. The middle of the span for a single-span shaft (Fig. 7, a) and the end of the cantilever shaft console (Fig. 7, b) are taken as the point of reduction of the distributed and concentrated masses. In this case, the shape of the axis of the curved shaft is considered to coincide with the shape of the axis with static bending of the transverse force applied at the point of reduction B.

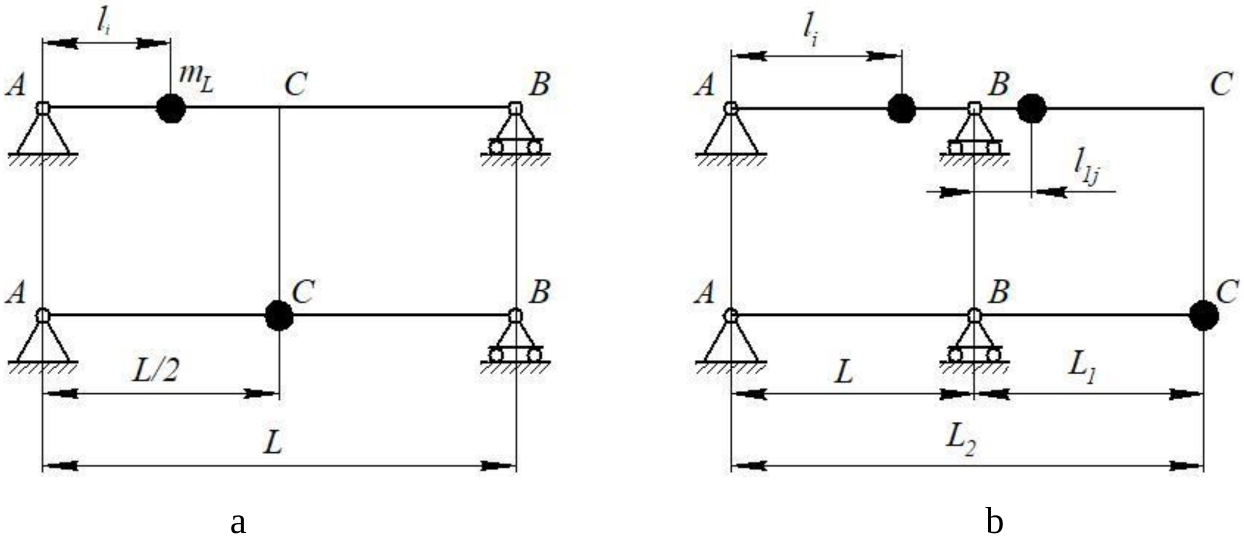


Fig. 7 Schematic diagrams of the drive shafts MMD: a - single-span; b – cantilever

9.1.6 Methods of engineering calculations of smooth shafts of MMD taking into account their own distributed masses

Calculations of shafts of mixing devices for vibration resistance are regulated by the guiding technical materials of the Ministry of Hydrofuels [482]. In the methods of calculating the shafts under consideration take a number of assumptions.

1. A split shaft connected by a rigid coupling is equivalent to a whole.

2. The force impact on the shaft of sealing devices and the pliability of the supports are not taken into account.

3. Points of application of masses and inertial forces from stirrers and other parts mounted on the shaft, adopted located in the middle of the hubs of these parts. If the parts have several hubs, its mass must be divided by the number of hubs.

4. Areas of the shaft located above the upper support are not taken into account in the calculation. In this case, the upper cantilever departure should not exceed 30% of the length of the cantilever shaft. The mass of the part mounted on it must not exceed the largest of the masses installed in the span or on the console, respectively.

5. Couplings and shaft diameter changes within the drive provided by standart.

6. When the agitator shaft is rigidly connected to the low-speed shaft of a standard planetary gearbox, the bearing of the gearbox used as the support of the agitator shaft is rigid.

7. Keyways up to 0.1d wide, where d is the diameter of the shaft, and local annular grooves with a diameter of more than 0.9d in the calculations for vibration resistance are not taken into account.

8. On rigid shafts of a constant section the sites differing in diameter no more than on 5% are allowed. The calculation is based on the diameter of the greatest length.

The calculation of the shaft for vibration resistance is performed by the method of successive approximations. The problem is reduced to the calculation of the approximate value of the diameter of the vibration resistance of the shaft, circular (angular) frequencies of its own transverse oscillations in the air and verify the result on the conditions of vibration resistance, given in Table 3.

Table 3 Vibration resistance conditions of shafts

Mixing medium	Vibration resistance conditions of shafts		
	Rigid		flexible
	with stirrers of all types, except paddle	with paddle stirrers	with high-speed stirrers
1	2	3	4
Air	$\xi = \frac{\omega}{\omega_{CR}} \le 0,7$	$\xi \le \frac{\omega}{\omega_{CR}} \le 0,7$	Not recommended
Fluid-fluid Fluid-solid	$\xi = \frac{\omega}{\omega_{CR}} \le 0,7$	$\xi = \frac{\omega}{\omega_{CR}} \le 0,7$ $\xi = \frac{\omega}{\omega_{CR}} \ne 0,45...0,55$	$\xi = \frac{\omega}{\omega_{CR}} = 1,3...1,6$
Fluid-air	$\xi = \frac{\omega}{\omega_{CR}} \le 0,6$	$\xi = \frac{\omega}{\omega_{CR}} \le 0,4$	Not recommended

The following restrictions are required when designing a flexible shaft:

a) Only flexible cantilever shafts of constant cross-section are used;

b) The use of flexible shafts in gas - liquid environments is not allowed;

c) There must be no detachable couplings within the calculated shaft length.

9.1.6.1 Instructions for calculating the vibration resistance of rigid smooth single-span and rigid and flexible smooth cantilever shafts

In accordance with [483], the first critical speed of smooth shafts is determined by the formulas:

a) For cantilever

$$\omega_{CR1} = \frac{\alpha_1^2}{L_1^2}\sqrt{\frac{E \cdot I}{m_L}}\ ; \tag{14}$$

б) For single-span

$$\omega_{CR1} = \frac{\alpha_1^2}{L^2}\sqrt{\frac{E \cdot I}{m_L}}\ , \tag{15}$$

Where α_1 is the first root of frequency equation (13); L_1 - the length of the console; $I=\pi d^4/64$ - moment of inertia of the shaft cross section (in the design calculation contains an unknown shaft diameter); m_L is the mass per unit length of the shaft; E is themodulus of elasticity of the shaft material; L is the span length (see Fig. 7).

For practical calculations of shafts, adequate to the accepted schematics (see Fig. 7), the values of α_1 obtained by numerical solution of differential equation (13) on a computer at given values of the parameters included in it: relative lengths of sections, relative reduced mass of shaft and planted details. The results of these calculations are given in the form of graphical dependences α [484]: for cantilever smooth shafts (Fig. 8) from the relative span length $\overline{L}$ and the relative reduced mass $\overline{m}_{red}$; for a single-span shaft (Fig. 9) from the relative reduced mass of parts $\overline{m}_{red}$.

The relative length of the cantilever shaft $\overline{L}$ is the ratio $\overline{L}/L_1$,, and the relative length of the cantilever $\overline{L}_1$ is the ratio $\overline{L}/L_1$. Here L_1, L_2, L – the length of the console, cantilever shaft and span, respectively (see Fig. 7).

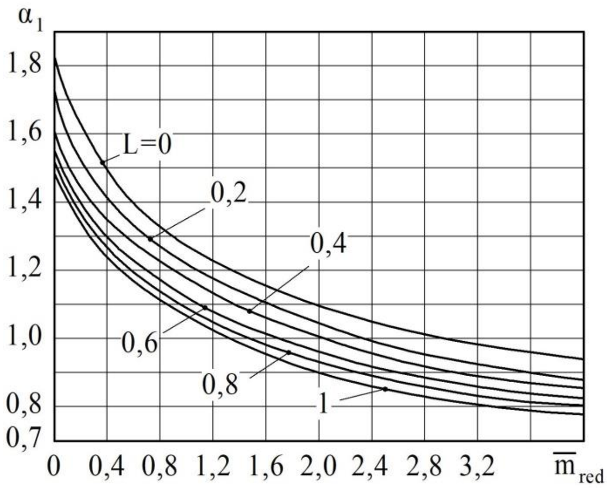


Fig. 8 Roots of the common equation for the cantilever shaft

The relative weight of the part:

For cantilever shafts

$$\overline{m}_{red} = \frac{m_{red}}{m_L \cdot L_1}; \tag{16}$$

For single-span shafts

$$\overline{m}_{red} = \frac{m_{red}}{m_L \cdot L}, \tag{17}$$

Where m_{red} total reduced mass of parts on the shaft; m_L is the linear mass of the shaft, $m_L = \pi d^2/\rho$ / ρ (d is the diameter of the shaft, ρ is the density of the shaft material).

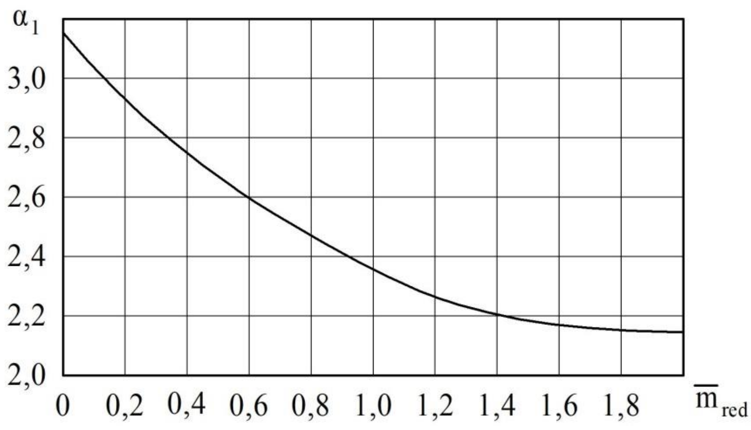


Fig. 9 Roots of the frequency equation for a single-span shaft

The total weight of the parts fixed in the following way:

On the cantilever shaft

$$m_{red} = \sum_{i=1}^{n_d} m_{ired} + \sum_{j=1}^{n_{1d}} m_{1jred}; \tag{18}$$

On the single-span shaft

$$m_{red} = \sum_{i=1}^{n_d} m_{ired} \quad , \tag{19}$$

Where n_d is the number of parts in the span; m_{ired} is the reduced mass of each i-th detail in the span; n_{1d} is the number of parts on the console; m_{1jred} is the reduced mass of each j-th part on the console.

Reduced mass:

Every j-th detail on the console

$$m_{1jred} = m_{1j} \cdot \bar{y}_{l1j}^2 ; \tag{20}$$

Every i-th in the span

$$m_{1red} = m_i \cdot \bar{y}_{li}^2 . \tag{21}$$

Dimensionless dynamic deflections $\bar{y}_{li}$ for a single-span shaft are determined from the graph (Fig. 10) as a function of the relative coordinates l_i of the centers of mass of the parts located on the shaft.

Dimensionless dynamic deflections y_{li} at the attachment points of parts, such as couplings in cantilever shaft spans:

$$y_{li} = K_i \cdot \bar{L} , \tag{22}$$

Where K_i is the coefficient determined from the graph (Fig. 11) as a function of $\bar{L} \cdot l_1$.

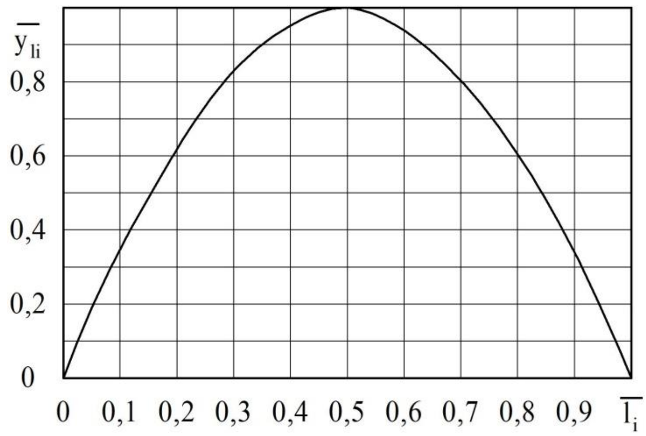


Fig. 10 Graph to determine the dimensionless deflections $\bar{y}_{li} = f(\bar{l}_i)$ of a single-span shaft

Dimensionless dynamic deflections $\bar{y}_{l1j}$ at the points of attachment of parts (stirrers) on the consoles of cantilever shafts are determined from the graphs (Fig. 12) as a function $\bar{L}$ and relative coordinates $\bar{l}_{1j}$ of the centers of mass of parts placed on the consoles.

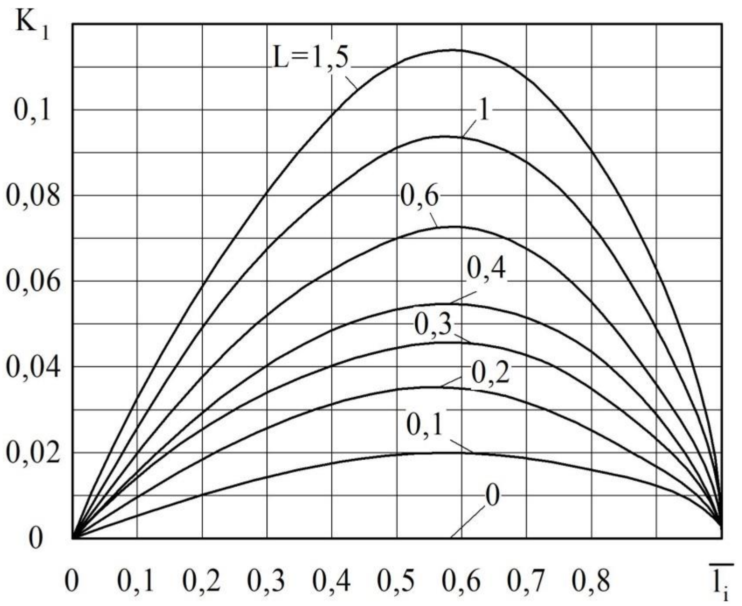


Fig. 11 Graph to determine the coefficient $K_i = f(\bar{l}_i, \bar{L})$

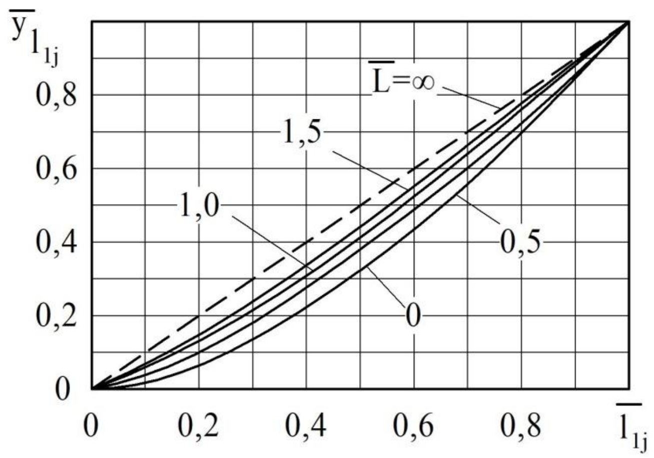


Fig. 12 Graph to determine the dimensionless deflections $\bar{y}_{l1j} = f(\bar{l}_{1j})$ on the cantilever shaft console

Relative coordinates $\bar{l}_i$ and $\bar{l}_{1j}$:

For single-span shafts and spans of cantilever shafts:

$$\bar{l}_i = \frac{l_i}{L}; \tag{23}$$

For consoles of cantilever shafts

$$\bar{l}_{ij} = \frac{l_{ij}}{L_1}. \tag{24}$$

In [485] the calculation formula is given

$$\bar{l}_{li} = \sin\left(\frac{\pi \cdot l_i}{L}\right), \tag{25}$$

Which is true for a single-span smooth shaft.

To determine the moment of inertia of the cross section of the shaft included in (14), (15), it is necessary to estimate the diameter of the vibration-resistant shaft. Using the method of reduction, we write for the selected schemes (see Fig. 7) the value of the total reduced mass M_{red} shaft and the parts attached to it:

$$\begin{cases} M_{red} = m_{red} + q \cdot m_L \cdot L_1; \\ M_{red} = m_{red} + q \cdot m_L \cdot L_2. \end{cases}; \tag{26}$$

Where m_{red} is the total reduced mass of parts (mixing devices) that are mounted on the shaft; q is the reduction factor of the shaft mass m_L; L_1, L_2 are the length of the span and cantilever shafts, respectively.

It is calculated that the real shaft (shaft with mass and parts attached to it) can be replaced by its equivalent weightless shaft with concentrated reduced mass.

Derivation of the formula for determining the approximate diameter of the shaft from the condition of equality of the critical speeds of the real and weightless shafts will be shown by the example of a single span shaft (Fig. 13):

$$\omega_{CR1a} = \omega_{CR1b} \tag{27}$$

Where ω_{CR1a} is the critical angular velocity of the real shaft; ω_{CR1b} is the critical angular velocity of the weightless shaft.

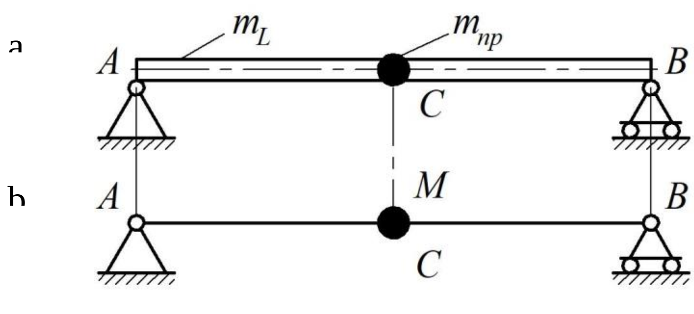


Fig. 13. Scheme for calculating the approximate value of the diameter of the vibration-resistant shaft: a - diagram of the shaft with distributed m_L and concentrated reduced m_{red} masses; b is a diagram of a weightless shaft with reduced mass.

In accordance with formulas (2), (26) and Table 1

$$\omega_{CR1} = \sqrt{\frac{1}{(M_{CR} \cdot \delta_{11})}} = \sqrt{\frac{3E \cdot I \cdot L}{l_1^2 (L - l_1)^2 (m_{red} + q \cdot m_L \cdot L)}}.$$

Since the midpoint of the shaft is chosen as the point of reduction C, then

$$\omega_{CRc} = \sqrt{\frac{16 \cdot 3E \cdot I \cdot L}{L^4 (m_{red} + q \cdot m_L \cdot L)}} = \sqrt{\frac{48E \cdot I}{L^3 (m_{red} + q \cdot m_L \cdot L)}}. \quad (28)$$

Given (15), (28), we represent equality (27) as

$$\frac{\alpha_{1a}^2}{L^2} \sqrt{\frac{E \cdot I}{m_L}} = \sqrt{\frac{48}{m_L \cdot L^3 \left(q + \frac{m_{red}}{m_L \cdot L} \right)}},$$

Whereof

$$\alpha_{1a}^2 = \sqrt{\frac{48 m_L \cdot L}{m_{red} + q \cdot m_L \cdot L}} = \sqrt{\frac{48}{m_L \cdot L^3 \left(q + \frac{m_{red}}{m_L \cdot L} \right)}}, \quad (29)$$

Where α_{1a} is the root of the frequency equation for a real shaft.

Using (15), (29), determine the approximate value of the first critical velocity of the real shaft

$$\omega_{CR1a} = \sqrt{\frac{48}{\frac{m_{red}}{m_L \cdot L} + q}} \cdot \sqrt{\frac{E \cdot I}{m_L \cdot L^4}} = \sqrt{\frac{48E \cdot I}{m_L \cdot L^4 \left(q + \frac{m_{red}}{m_L \cdot L} \right)}}. \quad (30)$$

Similarly, a formula can be obtained to calculate the approximate values of the critical angular velocities of the cantilever shafts.

Equating (30) to the critical speed from the condition of vibration resistance (see Table 3), we obtain the dependence for calculating the approximate value of the diameter of a smooth cantilever or single-span shaft:

$$\sqrt{\frac{\eta \cdot E \cdot I}{m_L \cdot L_y^2 \left(q + \frac{m_{red}}{m_L \cdot L} \right)}} = \frac{\omega}{\xi}, \quad (31)$$

Where η is the dimensionless coefficient that takes into account the conditions of shaft mounting; L_y - conditional length of the shaft; ω is the operating speed of the real shaft, which is set by the mixing conditions; ξ is a coefficient that takes into account the conditions of vibration resistance of the shafts (see Table 3).

For cantilever shafts $L_y = L_1$; $\eta = 3\bar{L}$; $q=0,25$.

For single span shafts $L_y = L$; $\eta=48$; $q=0,5$.

After transformations (31) we obtain a formula for calculating the approximate value of the diameter of the vibration-resistant rotor

$$d_p = \sqrt{q \cdot f \cdot L_y^2 + \sqrt{\left(q \cdot f \cdot L_y^2\right)^2 + \frac{8m_{red} \cdot L \cdot f}{\pi \cdot \rho}}} \ ,$$

Where f is the dimensionless coefficient that takes into account the vibration resistance of the shaft.

$$f = \frac{8\rho\omega^2 L_y^2}{\xi^2 \eta E} .$$

According to the method of engineering calculations [486]

$$d_p = \sqrt{A_1 + \sqrt{A_1^2 + A_2}} \ , \tag{32}$$

For one span shaft

$$A_1 = qfL^2 \ , \ A_2 = \frac{8m_{red} \cdot L \cdot f}{\pi \cdot \rho} .$$

For cantilever shaft

$$A_1 = qfL_1^2 \ , \ A_2 = \frac{8m_{red} \cdot L_1 \cdot f}{\pi \cdot \rho} .$$

Knowing the approximate value of the shaft diameter d_p, by formulas (14), (15) we determine the first critical velocity of the real shaft, which must satisfy the condition of vibration resistance (see Table 3) [487, 488]. If this condition is not met, the shaft diameter must be increased by 1 mm and find a new value of ω_{CR1}.

9.1.6.2 Instructions for calculating the vibration resistance of rigid stepped cantilever shafts

To reduce the mass of the shaft and some increase in the first critical angular velocity ω_{CR1} cantilever shafts should be designed stepped. The shape of the stepped shaft should be close to the shape of the beam of equal bending resistance (Fig. 14) [489].

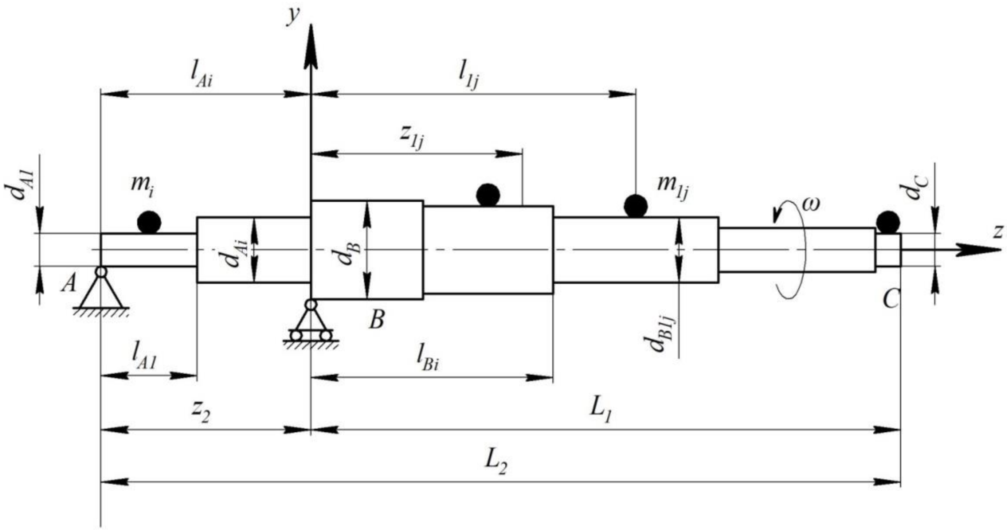


Fig. 14 The calculation scheme of cantilever stepped shaft with concentrated masses m_{1j} on the cantilever and m_i in the span of the shaft

The calculation of the vibration resistance of the stepped shaft can be divided into three stages:

1. Approximate calculation of the diameter of the vibration-resistant shaft in the support section B;

2. Calculation of diameters of steps of a shaft in span and on the console;

3. Calculation of the critical angular velocity of rotation of the real shaft and verification of the vibration resistance condition.

The calculation of the diameter d_B is performed by formulas (16), (18) - (24), (31) and (32).

The diameters d_{A1} and d_C are calculated from the torsional strength condition.

The shaft shank with a diameter d_{A1} perceives the torque spent on overcoming the friction forces in the supports A and C, in the seal and on the mixing of the liquid n_{1a} by the number of stirrers [490, 491].

The shank with a diameter of d_C transmits the torque required for mixing with only one stirrer:

$$M_T^C = \frac{N^C}{\omega},$$

Where N^C is the power consumed by the stirrer fixed at point C; ω is the operating angular velocity of the stirrer.

Since the polar moment of the shank resistance under consideration, $w_p^C=\pi d_C^3/16=M_{\kappa p}^C/[\tau]$ ([τ] - allowable tangential stress), the diameter of the shank C:

$$d_C = \sqrt{\frac{16N^C}{\pi \cdot \omega \cdot [\tau]}} . \tag{33}$$

When calculating the total torque transmitted by the shank d_{A1}, the moment of friction in the supports can be neglected due to its insignificant value.

Torque to overcome the friction forces in the seal, such as the stuffing box seal

$$M_T = \frac{F_f \cdot d_B}{2} = \left[\frac{\pi \cdot d_B \cdot S_H \cdot P \cdot f_{kin}}{2K \cdot f}\right]\left(e^{\frac{2Kfh}{S_H}} - 1\right)\frac{d_B}{2},$$

Where d_B is the diameter of the shaft in the support section; S_H, h - respectively, the length and height of the stuffing box seal; P - pressure in the device; f , f_{kin} - respectively static and kinetic coefficients of sliding friction of the shaft on the packing; K is the coefficient of lateral pressure.

Shank diameter A is measured similarly (33):

$$d_{A1} = \sqrt{\frac{16}{\pi \cdot [\tau]}\left(\sum_{n_{1d}=1}^{n_{1d}} \frac{N_{1d}}{\omega + M_T}\right)},$$

Where n_{1d} is the number of stirrers on the console; N_{n1d} is the power expended on mixing with each stirrer.

Diameters d_{A2}, d_{A3} are accepted taking into account fastening on a shaft of these or those details according to the set conditions.

The diameter of the steps on the console is chosen by the formula

$$d_{C1i} = d_C \sqrt{1 - \left(1 - \frac{d_C^2}{d_{A1}^2}\right) \cdot \overline{Z}_{1i}} \ ,$$

Where d_{A1} and d_C are the diameters of the shaft shanks; $\overline{z_{1\iota}} = z_{1i}/L_1$ is a relative coordinate of the end of the span; t is an indicator of the degree determined by the graph (Fig. 15) depending on the relative reduced mass of the parts mounted on the shaft m_{red} and the relative ductility of the parts θ:

$$\theta = \frac{I_C \cdot L}{I_{equ} \cdot L_1},$$

Where $I_C=\pi d_C^4/64$ is an axial moment of inertia of the reference section of the shaft; I_{equ} is the equivalent moment of inertia of the span cross section.

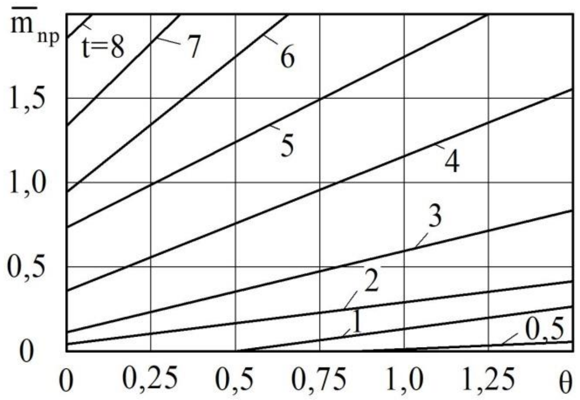


Fig. 15 Dependence of the exponent *t* on the relative mass of the parts mounted on the shaft and the relative flexibility θ span of the cantilever shaft

Depending on the number of steps in the span, its equivalent moment of inertia:

For a shaft with two steps d_{A1} and d_{A2}

$$I_{equ} = \frac{I_{A1}}{\frac{I_{A1}}{I_{A2}} + \left(\frac{l_{A1}}{L}\right)^3 \cdot \left(1 - \frac{I_{A1}}{I_{A2}}\right)},$$

For a shaft with three steps d_{A1}, d_{A2} and d_{A3}

$$I_{equ} = \frac{I_{A1}}{\frac{I_{A1}}{I_{A3}} + \left(\frac{l_{A1}}{L}\right)^3 \cdot \left(1 - \frac{I_{A1}}{I_{A2}}\right) + \left(\frac{l_{A2}}{L}\right)^3 \cdot \left(\frac{I_{A1}}{I_{A3}} - \frac{I_{A1}}{I_{A3}}\right)},$$

Where $I_{Ai}=\pi d_{Ai}^4/64$ is an axial moment of inertia of the cross section of the span step.

The first critical angular velocity of a real stepped cantilever shaft

$$\omega_{CR1} = \left[\frac{\overline{\omega}_{CR1} \cdot d_C}{4L_2^2}\right]\sqrt{\frac{E}{m_L}},$$

Where $\overline{\omega_{CR1}}$ is the relative first critical angular velocity of the real shaft; L_2 is the length of the cantilever shaft; E is the modulus of elasticity of the shaft material; m_L is mass of 1 m of shaft length.

Relative first critical angular velocity

$$\overline{\omega}_{CR1} = \sqrt{\frac{\overline{K}_{red}}{\overline{m}^{r}_{red}}},$$

Where $\overline{K_{red}}$ is the relative reduced stiffness of the stepped cantilever shaft; $\overline{m^{r}_{red}}$ is the relative reduced mass of the real shaft (stepped shaft with mounted parts).

Relative given stiffness factor

$$\overline{K}_{red} = b_3^2\left[b_2^2 + \frac{1}{3} - 2b_1\left(\frac{1}{t+1} - \frac{2}{t+2} + \frac{1}{t+3}\right) + b_1^2\left(\frac{1}{2t+1} - \frac{2}{2t+2} + \frac{1}{2t+3}\right)\right],$$

Relative reduced mass of the real shaft

$$\overline{m}^{r}_{red} = \overline{m}_{red} + b_3^2\left[\frac{b_2^2}{3} + \frac{11}{60}b_2 + \frac{11}{420} - b_1\left(\frac{b_2^2}{t+3} - \frac{b_2}{t+4} + \frac{1}{4(t+5)} - \frac{b_2}{3(t+5)} - \frac{1}{6(t+6)} + \frac{1}{36(t+7)}\right)\right]$$

Where $\overline{m_{red}}$ is the relative reduced mass of the parts fixed in the span and on the shaft console; b_1, b_2, b_3 are coefficients that take into account the shape and size of the stepped shaft

$$b_1 = 1 - \left(\frac{d_B}{d_C}\right) \quad ; \; b_2 = \frac{L \cdot I_C}{3L_1 \cdot I_{equ}} \; ; b_3 = \frac{1}{b_2 + \frac{1}{3}}.$$

Vibration resistance condition for rigid cantilever shaft

$$\frac{\omega}{\omega_{CR1}} = \xi ,$$

Where ξ is the coefficient selected from Table 3.

If the vibration resistance condition is not met, it is necessary to increase the diameter d_C and repeat the calculation until the vibration resistance condition is met.

CONCLUSIONS

The paper describes in detail the method and algorithm of typical calculation of shafts of mixing devices used in the chemical, food and biotechnological industries. The following algorithms and calculation methods are considered: calculation of the frequency of natural oscillations of a weightless shaft with one degree of freedom; instructions for calculating the critical angular velocity are given and the condition of vibration resistance of a smooth weightless shaft with one degree of freedom is established; instructions for calculating the frequency of natural oscillations of a smooth weightless shaft with two or more degrees of freedom; instructions for calculating the frequency of natural oscillations of a smooth shaft, taking into account its evenly distributed mass along the length; the method of engineering calculations of smooth shafts taking into account their own distributed masses is given. This work can be used as a textbook for university students with relevant engineering specialties in the field of mechanical engineering, and as a guide for engineering calculations in the design of relevant equipment in various industries.